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MARTIN BOUNDARIES OF RANDOM WALKS ON FUCHSIAN GROUPS

BY
CAROLINE SERIES

ABSTRACT

Let Γ be a finitely generated non-elementary Fuchsian group, and let μ be a probability measure with finite support on Γ such that supp μ generates Γ as a semigroup. If Γ contains no parabolic elements we show that for all but a small number of co-compact Γ, the Martin boundary M of the random walk on Γ with distribution μ can be identified with the limit set Λ of Γ. If Γ has cusps, we prove that Γ can be deformed into a group Γ', abstractly isomorphic to Γ, such that M can be identified with Λ', the limit set of Γ'. Our method uses the identification of Λ with a certain set of infinite reduced words in the generators of Γ described in [15]. The harmonic measure ν (ν is the hitting distribution of random paths in Γ on Λ) is a Gibbs measure on this space of infinite words, and the Poisson boundary of Γ, μ can be identified with Λ, ν.

Introduction

Let Γ be a finitely generated Fuchsian group, that is, a discrete subgroup of the group G of conformal automorphisms of the unit disk $D = \{z \in \mathbf{C} : |z| < 1\}$. Let μ be a probability distribution on Γ with finite support such that supp μ generates Γ as a semigroup. The random walk on Γ starting at $0 \in D$ with distribution μ is the sequence of random variables $g_n 0 = x_1 \cdots x_n 0$, where x_i is a Γ valued random variable with distribution μ.

The *limit set* Λ of Γ is the set of accumulation points of an orbit Γz_0, $z_0 \in D$. Λ is independent of z_0. If Λ consists of more than two points, Γ is said to be *non-elementary* and it is well known that Γ is then non-amenable. It is also well known that the random walk is transient, for example see [4]; that is, the probability that a path ever returns to 0 is less than one. Moreover, by theorem 1.3 of [9], with probability one $\lim_{n\to\infty} x_1, \cdots x_n 0 = \xi \in \Lambda$. Define the measure ν on Λ as the hitting distribution $\nu(A) = \Pr(\lim_{n\to\infty} x_1 \cdots x_n 0 \in A)$ for $A \subseteq \Lambda$.

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A function h on Γ is called *μ-harmonic* if $h(x)=\Sigma_{g\in\Gamma}\,\mu(g)h(xg)$ for all $x\in\Gamma$. The Martin boundary M of Γ, μ is a compactification of Γ such that all harmonic functions have a representation

$$h(x)=\int_M K(x,\xi)d\lambda(\xi),\qquad x\in\Gamma,$$

where $K(x,\xi)$ is the so-called *Martin kernel* and λ is a finite measure on M. The object of this paper is to determine M.

If Γ contains no parabolic elements (an element is parabolic if it has exactly one fixed point on S^1), then we show that $M=\Lambda$. Otherwise, we show that by moving apart the cusps slightly Γ may be deformed to a group Γ' abstractly isomorphic to Γ, with generators satisfying the same relations as in Γ. M clearly only depends on Γ considered as an abstract group, so that in this case we obtain $M=\Lambda'$. Λ' is a covering of Λ, in the sense that there is a Γ-equivariant surjective map $\Lambda'\to\Lambda$ which fails to be injective only at parabolic points in Λ, where it is two to one.

This result is known when Γ is the free group on two generators F_2 [5], [3]. F_2 can be embedded in G as follows: take four disjoint circular arcs in D_2 orthogonal to the unit circle S^1, and let a, b be elements of G which carry the exterior of one arc to the interior of the opposite arc [15]. It is well known that $\langle a,b\rangle=F_2$. The limit set of F_2 is a Cantor set consisting of the intersection of all images of the regions inside the four circles. Λ can be identified with the space Σ of infinite reduced words in the generators a, b; that is words in which a generator is never followed by its inverse. In [5] and [3] the Martin boundary is identified with Σ. Contrary to the suggestion in [5], it is a non-trivial exercise to extend these results to the general case.

The modification in case Γ has cusps, referred to above, is illustrated in this case when the four defining circles described above touch on S^1, so that F_2 has cusps. The deformation consists in this case of replacing F_2 by the group F_2' obtained by slightly shrinking and separating the defining circles and making the same identifications as before.

Our proof of the general result uses the method of [15], in which it is shown that, for all but a small number of co-compact Γ with few generators and short relations, it is possible to construct a space of infinite reduced words Σ and a map $\pi:\Sigma\to\Lambda$ which is surjective and which fails to be injective only at a countable number of points at which it is two to one. Using this construction it is possible by elaborating the methods of [5] and [3] to identify the Martin boundary with Λ. An important step in the proof is a result which has independent interest, namely a Perron–Frobenius type theorem for random products of infinite positive matrices. This is proved in section 5. This theorem is not unlike the infinite dimensional version of the multiplicative ergodic theorem for compact operators in [13]. A finite dimensional version of the same result is given in [14].

The *harmonic measure* on $\Lambda = M$ is given by

$$\nu(E) = \Pr\left[\lim_{n\to\infty} x_1 \cdots x_n 0 \in E\right].$$

Intuitively ν is the hitting distribution on Λ of random paths starting at 0. Any bounded harmonic function on Γ has a representation

$$h(x) = \int_\Lambda K(x,\xi) f(\xi) d\nu(\xi),$$

where f is a bounded Borel function on Λ ([11] lemma 10.42). This representation is unique up to sets of ν measure zero. It follows from the results in [10] and [11] that Λ, ν is what Furstenberg calls the Poisson boundary of Γ, μ.

Now ν may be regarded as a measure on Σ. It was shown in [15] that Σ is of finite type [2]. We show in section 6 that ν is a Gibbs measure with respect to the function

$$\phi(e_1e_2\cdots) = -\log K(e_1, e_1e_2\cdots), \qquad e_1e_2\cdots \in \Sigma.$$

This allows one to deduce, for example, a central limit theorem for random sequences of the type described in [16], see [2].

In section 1 we recall the basic facts about Martin boundaries and state the main results of the paper. In section 2 we summarize necessary facts about Fuchsian groups and describe the relevant features of the space Σ of infinite reduced words referred to above. In sections 3 and 4 we establish the main technical result that the probability $u(x, y)$ that a random path starting at x ever reaches y can be expressed as a coefficient in a certain random product of infinite matrices.

Throughout, e will denote the identity of Γ. Without loss of generality we may assume that 0 is not a fixed point of Γ, so that the map $\Gamma \to \Gamma 0$ is injective. We shall write x for $x0$ when this does not lead to confusion. We always label arcs on S^1 in an anti-clockwise direction, so that $[P, Q]$ means the closed interval of points between P and Q moving in an anti-clockwise direction. For $P, Q \in S^1$ we write $|P - Q|$ for the Euclidean distance between P and Q. If v, w are vertices of $G(\Gamma)$ we write $d(v, w)$ for the number of edges in the shortest path in $G(\Gamma)$ from v to w and $|v|$ for $d(0, v)$. If $g \in \Gamma$ we write $|g| = d(0, g0)$.

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§1. The Martin boundary and harmonic functions

Let Γ be a finitely generated non-elementary Fuchsian group without cusps, and let μ be a probability distribution on Γ with finite support. Let $u(x, y)$, $x, y \in \Gamma$, be the probability that starting at x, a random path $x_1 \cdots x_n$ with distribution μ ever reaches y. The *Martin kernel* of the random walk is defined by

$$K(x, y) = \frac{u(x, y)}{u(e, y)}, \qquad x, y \in \Gamma.$$

Γ can be embedded as a subset of $\mathbf{R}^{\Gamma}$ by the map $i : y \to K(\cdot, y)$. Then $\overline{i(\Gamma)}$ is compact and the space $B = \overline{i(\Gamma)} - i(\Gamma)$ is called the Martin boundary of Γ, μ. For these and other results about Martin boundaries we refer the reader to [11] or for a less formal account to [6].

We shall show that $\overline{i(\Gamma)}$ can be identified with $\Gamma \cup \Lambda$, with the topology induced as a subset of $\bar{D}$. The main steps are the following two results:

THEOREM 1.1. *Let* $y_n \in \Gamma$, $n = 1, 2, \cdots$, *and let* $\lim_{n \to \infty} y_n = \eta \in \Lambda$. *Then for each* $x \in \Gamma$, $K(x, \eta) = \lim_{n \to \infty} K(x, y_n)$ *exists and is independent of* $\{y_n\}$.

THEOREM 1.2. *Let* $\xi \in \Lambda$ *and suppose* $e_1 e_2 \cdots \in \pi^{-1}(\xi)$. (For a precise definition of Σ, π, see below.) *Then*

(i) $$\lim_{z \to \eta} K(z, \xi) = 0 \qquad \text{for } \eta \in \Lambda, \quad \eta \neq \xi.$$

The convergence is uniform on some neighbourhood U of η, in the sense that for each $\varepsilon > 0$, there exists $\delta > 0$ such that $K(z, \xi) < \varepsilon$ whenever $|z - \eta'| < \delta$ for some $\eta' \in U$.

(ii) $$\lim_{n \to \infty} K(e_1 \cdots e_n, \xi) = +\infty.$$

COROLLARY 1.3. *The functions* $K(x, \cdot)$, $x \in \Gamma$, *separate points of* $\Gamma \cup \Lambda$.

PROOF. This follows from Theorem 1.2 except for the case $x, y \in \Gamma$, $x \neq y$. This is easily dealt with by noting that the equalities $K(x, y) = K(x, x)$ and $K(y, x) = K(y, y)$ cannot hold simultaneously, for this forces $u(x, y)u(y, x) = 1$, which contradicts transience of the random walk.

COROLLARY 1.4. *The Martin boundary of* Γ, μ *is* Λ.

PROOF. Define $T:\Gamma\cup\Lambda\to\overline{i(\Gamma)}$ as follows:

if $x\in\Gamma$, $T(x)=K(\cdot,x)$;

if $\xi\in\Lambda$ choose $e_1e_2\cdots\in\pi^{-1}(\xi)$. Since $\lim_{n\to\infty}e_1\cdots e_n0=\xi$ (see Section 2), by Theorem 1.1, $K(\cdot,\xi)$ exists as a pointwise limit of functions in $i(\Gamma)$. Define $T(\xi)=\lim_{n\to\infty}K(\cdot,e_1\cdots e_n)$. T is injective by Corollary 1.3.

If $h\in\overline{i(\Gamma)}$ then h is a pointwise limit of functions $K(\cdot,y_n)$. If $\lim_{n\to\infty}y_n=\eta$ exists then $h=T(\eta)$ by Theorem 1.1. Otherwise $\{y_n\}_{n=1}^{\infty}$ has at least two accumulation points $\xi,\eta\in\Gamma\cup\Lambda$. But then $K(\cdot,\xi)\neq K(\cdot,\eta)$ by Corollary 1.3, which is impossible.

HARMONIC FUNCTIONS. A function h on Γ is μ-harmonic if $h(x)=\Sigma_{g\in\Gamma}h(xg)\mu(g)$. Let H denote the set of non-negative μ-harmonic functions with $h(e)=1$. We recall some general facts:

(i) $K(\cdot,\xi)\in H$ for any $\xi\in\overline{i(\Gamma)}$.

(ii) Any extremal of H is of the form $K(\cdot,\xi)$, $\xi\in\overline{i(\Gamma)}-\Gamma$, where we have identified Γ with $i(\Gamma)$ using Corollary 1.3.

(iii) Each $h\in H$ has a representation

$$h(x)=\int_{B_e}K(x,\eta)dm(\eta),\qquad x\in\Gamma,$$

where B_e is the set of extremals of H and m is some probability measure on B_e.

COROLLARY 1.5. *The functions* $K(\cdot,\xi)$, $\xi\in\overline{i(\Gamma)}-\Gamma$, *are precisely the extremals of* H.

PROOF. Suppose

$$K(x,\xi)=\alpha h_1(x)+(1-\alpha)h_2(x),\qquad h_1,h_2\in H,\quad x\in\Gamma,\quad \xi\in\Lambda,\quad 0\leqq\alpha\leqq 1.$$

By (iii) we may write

$$K(x,\xi)=\int_{B_e}K(x,\eta)dm(\eta)$$

for some probability measure m on B_e; and by (ii) and Corollary 1.4 this is the same as

$$K(x,\xi)=\int_{\Lambda}K(x,\eta)dm(\eta).$$

Suppose $\eta_0\in\Lambda$, $\eta_0\neq\xi$. By Theorem 1.2 there is a neighbourhood V_0 of η_0 so that $K(g_n,\eta)\to 0$ uniformly on V_0, where $e_1e_2\cdots\in\pi^{-1}(\xi)$ and $g_n=e_1\cdots e_n$. Thus

$$\lim_{n\to\infty}\int_{V_0} K(g_n,\eta)dm(\eta)=0.$$

This forces m to be a point mass at ξ.

PARABOLIC ELEMENTS IN Γ. We shall show in the next section that if Γ has parabolic elements, we may replace Γ by an isomorphic group Γ' in such a way that $M = M'$. The above results then show that $M = \Lambda'$.

§2. Fuchsian groups

A Fuchsian group is a discrete subgroup of the group

$$G=\left\{\begin{pmatrix} a & b \\ \bar{b} & \bar{a}\end{pmatrix} : a,b\in C, |a|^2-|b|^2=1\right\}$$

of conformal automorphisms of the unit disk D. Elements of Γ are called parabolic, hyperbolic or elliptic according as they have one or two fixed points on S^1, or one fixed point in D. As in the introduction we may assume that 0 is not an elliptic fixed point. Parabolic fixed points of Γ are called cusps.

Any finitely generated group Γ has a fundamental polygon R which is a region bounded by a finite number of circular arcs orthogonal to S^1, such that no two interior points of R are conjugate under Γ and such that every point in D is conjugate to some point in $\bar{R}$. Each side s of $R\cap D$ is identified with another side s', by an element $g(s)\in\Gamma$. The set $\Gamma_0=\{g(s): s\subset\partial R\}$ forms a symmetrical set of generators for Γ ([8], §23). Γ has parabolic elements if and only if two of the circles bounding R touch at a point on S^1. Such a point is a cusp of Γ.

The graph $G(\Gamma)$ of Γ, with respect to the generators Γ_0, may be represented as a net in D. The vertices are the points $g0$, $g\in\Gamma$, and the edges are the directed lines joining vertices $g0$, $g'0$ whenever $g^{-1}g'\in\Gamma_0$. Such an edge we label $g^{-1}g'$. $G(\Gamma)$ is the dual graph to the graph formed by all the images of ∂R under Γ.

Relations in Γ correspond to closed paths in $G(\Gamma)$. Regions bounded by edges of $G(\Gamma)$, together possibly with arcs of S^1, with no edges intersecting the interior, we call polygons. The polygons which meet at 0 may have a finite or infinite number of sides. Thinking of $G(\Gamma)$ as the dual of the graph of images of ∂R, it is clear that the latter occurs only when $\partial R\cap S^1\neq\emptyset$. Let P be such an infinite sided polygon. One sees that $P\cap S^1=\{\xi\}$ if and only if ξ is a cusp of Γ. In this case one can deform the fundamental region R by slightly moving apart the two sides of R which touch at ξ. Doing this for all cusps of ∂R, we obtain a new fundamental region R'. If the same identification of sides is made as before, we obtain an identical set of generators Γ_0' satisfying the same relations as Γ_0 ([8],

[12]). This deformed group Γ' is the one referred to at the end of the previous section.

We shall call a pair Γ, Γ_0 obtained as above *non-exceptional* if one of the following conditions hold:

(i) D/Γ is not compact.

(ii) At least five edges meet at each vertex of $G(\Gamma)$ and every polygon in $G(\Gamma)$ has at least five sides.

(iii) At least three edges meet at each vertex of $G(\Gamma)$ and at least three of the polygons meeting at a vertex have at least seven sides.

There is a small number of non-elementary exceptional groups, see [15] figs. 4 and 5.

In the remarks that follow it is immaterial whether or not Γ has cusps.

In general an element $g \in \Gamma$ may be represented as a product of generators in many different ways. Among these some are shortest in the sense that they use the minimum possible number of generators. Clearly this shortest length is $|g|$. In [15] it is shown that for a non-exceptional group Γ, one can lay down certain simple rules, depending on the relations in Γ, in such a way that if $g \in \Gamma$ then g has a unique shortest representation as a product of generators satisfying these rules. Such a representation is called *admissible.* For example, if $e_1 \cdots e_{2k} = 1$ is a relation in Γ, one of the rules specifies that $e_1 \cdots e_k$ never occurs, being instead replaced by $e_{2k}^{-1} \cdots e_{k+1}^{-1}$. The precise details of the rules need not concern us here. Let $\Sigma_F = \{e_1 \cdots e_n : e_i \in \Gamma_0,\ e_1 \cdots e_n \text{ is admissible}\}$.

THEOREM 2.1 ([15] proposition 4.6 and theorem 4.10). *Let* $\Sigma = \{e_1 e_2 \cdots \in \Gamma^{\mathbb{N}} : e_1 \cdots e_n \in \Sigma_F$ *for each* $n\}$.

Then $\lim_{n\to\infty} e_1 \cdots e_n 0$ *exists for each* $w = e_1 e_2 \cdots \in \Sigma$. *The limit is uniform on* Λ. *The map* $\pi : \Sigma \to \Lambda$, $\pi(e_1 e_2 \cdots) = \lim_{n\to\infty} e_1 \cdots e_n 0$ *is a continuous surjection which is injective except at a countable number of points where it is two to one.*

We need some more facts about admissible sequences. For details we refer to [15].

Let $e_1 \cdots e_p \in \Sigma_F$. Suppose we follow a path starting at 0, along $e_1 0, e_1 e_2 0, \cdots, e_1 \cdots e_p 0$; and then at every succeeding vertex turn as far as possible to the left, subject only to the constraint that at each step the path followed be admissible. The portion of this path from $e_1 \cdots e_p$ to S^1 is called the extreme left path from $e_1 \cdots e_p$ and its endpoint in S^1 is denoted $\lambda(e_1 \cdots e_p)$. Likewise the extreme right path from $e_1 \cdots e_p$ turns at every step after $e_1 \cdots e_p 0$ as far as possible to the right subject to the constraint that it be admissible. This path has endpoint $\rho(e_1 \cdots e_p)$. For $e_1 \cdots e_p \in \Sigma_F$, let $I(e_1 \cdots e_p) =$

$[\rho(e_1\cdots e_p), \lambda(e_1\cdots e_p)]$. Let $Z(e_1\cdots e_p)=\{a_1a_2\cdots\in\Sigma : a_i=e_i,\ i\leqq p\}$. Then $I(e_1\cdots e_p)\supseteq I(e_1\cdots e_{p+1})$ for each p, $|I(e_1\cdots e_p)|\to 0$ as $p\to\infty$, $\pi(Z(e_1\cdots e_p))\subseteq I(e_1\cdots e_p)$ and if $\xi=\pi(e_1e_2\cdots)$ then $\{\xi\}=\bigcap_{p=1}^{\infty} I(e_1\cdots e_p)$.

If $e_1\cdots e_p\neq f_1\cdots f_p$ then $\operatorname{Int} I(e_1\cdots e_p)\cap \operatorname{Int} I(f_1\cdots f_p)=\emptyset$. We shall call any interval $I(e_1\cdots e_p)$, $e_1\cdots e_p\in\Sigma_F$, an interval of rank p. For fixed $e_1\cdots e_p$, consider all the intervals $I(e_1\cdots e_pa)$ of rank $p+1$. Using the fact that at least three edges meet at each vertex of $G(\Gamma)$ (since Γ is non-elementary) and the rules described in [15] it is easy to see that there are always at least two such intervals.

2.2. *The Finiteness Property*

The admissibility rules satisfy a certain finiteness property which is crucial. Namely there are only a finite number of distinct intervals $(e_1\cdots e_{p-1})^{-1}I(e_1\cdots e_{p-1}a)$, $a\in\Gamma_0$, $p\geqq 2$, $e_1\cdots e_p\in\Sigma_F$, [15], theorem 4.12.

This property is used in [15] to prove that the space Σ can be coded to a subshift of finite type [2].

2.3. *Hyperbolic Distance*

Distances in D can be measured either in the usual Euclidean sense or hyperbolically. Two points $x,y\in D$ lie on a unique circle γ orthogonal to the boundary S^1. The hyperbolic distance of x and y is given by the formula $h(x,y)=|\log[x,y,P,P']|$, where P, P' are the points where γ cuts S^1 and $[\cdot,\cdot,\cdot,\cdot]$ denotes cross ratio. Hyperbolic distance is invariant under conformal automorphisms of D. Hyperbolic and Euclidean distance are related by the formula

$$d_{\mathrm{Eucl.}}(0,x)=\tanh \tfrac{1}{2}h(0,x).$$

Suppose Γ, Γ_0 are given as above. Provided Γ has no cusps, there are constants α, $\beta>0$ so that

$$\alpha\,|g|<h(0,g0)<\beta\,|g|\qquad ([7], §4).$$

One deduces that there are $\gamma,\delta>0$ so that

$$d_{\mathrm{Eucl.}}(x,y)<\gamma e^{-\delta n}\qquad \text{for } x,y\in\Gamma$$

whenever $x=g_1\cdots g_s\in\Sigma_F$, $y=f_1\cdots f_t\in\Sigma_F$ and $e_i=f_i$ for $i\leqq n$ ([15], §4).

§3. Convergence of random paths

Throughout the next three sections Γ is a finitely generated non-exceptional Fuchsian group without cusps and Γ_0 a set of generators obtained from a fundamental region as described above. We suppose that μ is probability distribution of finite support on Γ.

By [9] theorem 1.3 the random walk on Γ with distribution μ is such that $\lim_{n\to\infty} x_1\cdots x_n 0$ exists with probability one. In particular the random walk is transient. Obviously, $\lim_{n\to\infty} x_1\cdots x_n 0\in\Lambda$, the limit set of Γ. To compute the Martin boundary, the main problem is to obtain a tractable expression for the probability $u(x,y)$ that a random path starting at $x0$ ever reaches $y0$. To this end we use the description of Λ outlined in section 2.

We begin by assuming that $\mu(g)>0$ if and only if $g\in\Gamma_0$ and indicate briefly at the end how to modify the construction in the general case.

Let $D^*=D-\bigcup\{P: P$ is an open region enclosed by an infinite sided polygon in $G(\Gamma)\}$ and let $S^*=S^1\cap\overline{D^*}$. We shall find $m\in\mathbf{N}$, and for $p\geqq 1$ define regions $R(e_1\cdots e_p)\subset\bar{D}$, $e_1\cdots e_p\in\Sigma_F$, such that

(i) $I(e_1\cdots e_p)\subset S^1\cap R(e_1\cdots e_p)$,

(ii) $\mathrm{Int}(R(e_1\cdots e_p)\cap D^*)\supset R(e_1\cdots e_{p+m})\cap D^*$ whenever $e_1\cdots e_{p+m}\in\Sigma_F$, and

(iii) $\mathrm{Int}(R(e_1\cdots e_p)\cap S^*)\supset R(e_1\cdots e_{p+m})\cap S^*$.

Case 1. $I(e_1\cdots e_p)\subset\mathrm{Int}\,I(e_1\cdots e_{p-1})$. Let $R(e_1\cdots e_p)$ be the closed region bounded by the right and left extreme paths from $e_1\cdots e_{p-1}0$ and $I(e_1\cdots e_{p-1})$.

Case 2. $I(e_1\cdots e_p)\cap\partial I(e_1\cdots e_{p-1})\neq\emptyset$. Suppose to fix notation that $\lambda(e_1\cdots e_p)\in I(e_1\cdots e_{p-1})$. Let $I(f_1\cdots f_p)$ be the interval of rank p next to $I(e_1\cdots e_p)$ in anti-clockwise order round S^1.

Case 2a. $I(e_1\cdots e_p)\cap I(f_1\cdots f_p)=\emptyset$. Let $R(e_1\cdots e_p)$ be the closed region bounded by the right and left extreme paths from $e_1\cdots e_{p-1}0$ and $I(e_1\cdots e_{p-1})$.

Case 2b. $I(e_1\cdots e_p)\cap I(f_1\cdots f_p)\neq\emptyset$. Let σ, τ denote the extreme left and right paths from $e_1\cdots e_{p-1}0$ and $f_1\cdots f_{p-1}0$ respectively. Then there are vertices $v\in\sigma$, $w\in\tau$ which are joined by a single edge of $G(\Gamma)$. We prove this as follows: let S be the region bounded by the path $0, e_10,\cdots, e_1\cdots e_{p-1}0, \sigma$, and the path $0, f_10,\cdots, f_1\cdots f_{p-1}0,\tau$. There can be no vertices in $\mathrm{Int}\,S$. For let ξ be the endpoint of one of the extreme paths from such a vertex. Since admissible paths cannot meet in D, $\xi\in S$. Suppose $g_1g_2\cdots\in\pi^{-1}(\xi)$. Then $I(g_1\cdots g_p)$ would be an interval of rank p lying between $I(e_1\cdots e_p)$ and $I(f_1\cdots f_p)$, which is impossible. Now some pair $v\in\sigma$, $w\in\tau$ must be joined by a path in S because otherwise there would be an infinite sided polygon in $G(\Gamma)$, with edges meeting on S^1, contradicting the assumption that Γ has no cusps. By the above remarks, this path must consist of a single edge E.

Choose $v \in \sigma$ and $w \in \tau$ to be as close as possible to $e_1 \cdots e_{p-1}0$ and $f_1 \cdots f_{p-1}0$. Let $R(e_1 \cdots e_p)$ be the region bounded by the left extreme path from $f_1 \cdots f_{p-1}0$, the right extreme path from $e_1 \cdots e_{p-1}0$; the paths in σ, τ joining $e_1 \cdots e_{p-1}0$ to v and $f_1 \cdots f_{p-1}0$ to w; E and $I(e_1 \cdots e_{p-1}) \cup I(f_1 \cdots f_{p-1})$.

Notice that $d(v, e_1 \cdots e_{p-1})$, $d(w, e_1 \cdots e_{p-1}) \leqq m_0 - 1$, where m_0 is the maximum number of sides of a finite sided polygon in $G(\Gamma)$.

PROPOSITION 3.1. *There exists* $m \in \mathbf{N}$ *so that*
$S^* \cap R(e_1 \cdots e_{p+m}) \subset \mathrm{Int}(S^* \cap R(e_1 \cdots e_p))$ *whenever* $p \geqq 1$, $e_1 \cdots e_{p+m} \in \Sigma_F$.

PROOF. Suppose first $I(e_1 \cdots e_p) \subset \mathrm{Int}\, I(e_1 \cdots e_{p-1})$. Then for $m \geqq 2$, $S^1 \cap R(e_1 \cdots e_{p+m})$ is an interval containing one or two adjacent intervals of rank $p+m-1$, at least one of which is contained in $I(e_1 \cdots e_p)$. On each side of $I(e_1 \cdots e_p)$ is at least one interval $I(e_1 \cdots e_{p-1}a) \subset I(e_1 \cdots e_{p-1})$, $a \in \Gamma_0$. Each $I(e_1 \cdots e_{p-1}a)$ contains at least two intervals of rank $p+m-1$. Thus $S^1 \cap R(e_1 \cdots e_{p+m})$ is bounded away from the ends of $I(e_1 \cdots e_{p-1})$.

Suppose on the other hand that $I(e_1 \cdots e_p) \cap \partial I(e_1 \cdots e_{p-1}) \neq \emptyset$, say $\lambda(e_1 \cdots e_p) \in \partial I(e_1 \cdots e_{p-1})$. Again $S^1 \cap R(e_1 \cdots e_{p+m})$, $m \geqq 2$, is an interval containing one or two adjacent intervals of rank $p+m-1$, at least one lying in $I(e_1 \cdots e_p)$.

Suppose we are in case 2b so that $R(e_1 \cdots e_p) \cap S^1 = I(e_1 \cdots e_{p-1}) \cup I(f_1 \cdots f_{p-1})$, where $I(e_1 \cdots e_{p-1}) \cap I(f_1 \cdots f_{p-1}) = \{\lambda(e_1 \cdots e_{p-1})\}$. Then $I(e_1 \cdots e_p) \subset I(e_1 \cdots e_{p-1})$ and $I(e_1 \cdots e_p) \subset \mathrm{Int}(R(e_1 \cdots e_p) \cap S^1)$. On each side of $I(e_1 \cdots e_p)$ there is at least one interval of rank p contained in $R(e_1 \cdots e_p) \cap S^1$, which gives the result.

In case 2a we have $R(e_1 \cdots e_p) \cap S^1 = I(e_1 \cdots e_{p-1})$, and $I(e_1 \cdots e_{p-1}) \cap I(f_1 \cdots f_{p-1}) = \emptyset$. Let σ, τ be the left and right extreme paths from $e_1 \cdots e_{p-1}0$, $f_1 \cdots f_{p-1}0$ respectively. Only a finite number of vertices $v \in \sigma$, $w \in \tau$ can be joined by a path in $G(\Gamma)$, for arguing as before such a path can have only one edge and so an infinite number of joining edges would imply that σ, τ have the same end points. There is a bound m_1 independent of p on $d(v, e_1 \cdots e_{p-1})$ and $d(w, f_1 \cdots f_{p-1})$ for such vertices v, w; for if any such v, w exist then $d(f_1 \cdots f_{p-1}, e_1 \cdots e_{p-1}) \leqq m_0$ and there are only a finite number of possible arrangements of the polygons in $G(\Gamma)$. Choosing $m > m_1$ the argument is completed by observing that as we are working in S^* we now need only consider the right endpoint $\rho(e_1 \cdots e_{p-1})$ of $I(e_1 \cdots e_{p-1})$, and this works as in case 2b above.

COROLLARY 3.2. $D^* \cap R(e_1 \cdots e_{p+m}) \subset \mathrm{Int}\, D^* \cap R(e_1 \cdots e_p)$ *for* $e_1 \cdots e_{p+m} \in \Sigma_F$.

PROOF. This can only fail if some vertex $v \in R(e_1 \cdots e_{p+m})$ lies in $\partial R(e_1 \cdots e_p)$. Since $|v| \geqq p+m-1$, v must lie on one of the extreme paths in $\partial R(e_1 \cdots e_p)$ which converges to a point in $\partial(S^1 \cap R(e_1 \cdots e_p))$. In cases 1 and 2b above this is impossible because the part of this path from v to S^1 lies in $\partial R(e_1 \cdots e_{p+m})$. In case 2a the choice of m forces either $v \in D^*$, which is allowed, or the argument works as in case 1.

From now on, we fix m as above.

For $e_1 \cdots e_{mk} \in \Sigma_F$, $k \geqq 1$, define inductively:

$V(e_1 \cdots e_m) = \{v \in \partial R(e_1 \cdots e_m) : v$ can be reached from 0 by a path of positive probability which first cuts $\partial R(e_1 \cdots e_m)$ in $v\}$;

$V(e_1 \cdots e_{mk}) = \{v \in \partial R(e_1 \cdots e_{mk}) : v$ can be reached from $V(e_1 \cdots e_{m(k-1)})$ by a path of positive probability which first cuts $\partial R(e_1 \cdots e_{mk})$ in $v\}$.

PROPOSITION 3.3. *Let $\xi \in \Lambda$ and suppose $e_1 e_2 \cdots \in \pi^{-1}(\xi)$. Let $x_1 x_2 \cdots$ be a random path starting at 0 converging to ξ. Then there exist integers $0 < n_1 < n_2 < \cdots$ such that*

$$x_1 \cdots x_{n_k} \in V(e_1 \cdots e_{mk}) \qquad \text{for } k \geqq 1.$$

PROOF. For $k \geqq 1$, $x_1 \cdots x_n \in R(e_1 \cdots e_{mk})$, for large n. This is clear if $I(e_1 \cdots e_{mk}) \subset \operatorname{Int}(S^1 \cap R(e_1 \cdots e_{mk}))$. Otherwise $S^* \cap I(e_1 \cdots e_{mk}) \subset \operatorname{Int}(S^* \cap R(e_1 \cdots e_{mk}))$. Then there is a neighbourhood of $I(e_1 \cdots e_{mk})$ in $\bar{D}$ which contains no vertices other than those in $R(e_1 \cdots e_{mk})$, which gives the result.

Since $0 \in R(e_1 \cdots e_m)$ it follows from Corollary 3.2 that $x_1 x_2 \cdots$ must cross successively $\partial R(e_1 \cdots e_m), \partial R(e_1 \cdots e_{2m}), \cdots$. Moreover the sequence of first crossings $\{x_1 \cdots x_{n_k}\}$ satisfies $x_1 \cdots x_{n_k} \in V(e_1 \cdots e_{mk})$, and $0 < n_1 < \cdots < n_k < n_{k+1} < \cdots$ unless possibly if

$$w = x_1 \cdots x_{n_k} 0 \in \partial R(e_1 \cdots e_{mk}) \cap \partial R(e_1 \cdots e_{m(k+1)}) \qquad \text{for some } k,$$

which can only happen if $w \in \partial D^*$.

Now $V(e_1 \cdots e_{mk}) \cap \partial D^*$ consists of at most one point v which is at distance at most m_1 from $e_1 \cdots e_{mk-1}$. Since $w \in \partial R(e_1 \cdots e_{m(k+1)})$ we have $|w| \geqq m(k+1)$ so that $v = w$ is impossible.

Suppose now μ is any measure of finite support on Γ, and let $N = \max\{|g| : \mu(g) > 0\}$. We shall show that there exists $M > 1$ so that Proposition 3.3 holds with M replacing m.

LEMMA 3.4. *Let σ, τ be paths in $G(\Gamma)$ with endpoints ξ, η satisfying*

$|\xi - \eta| \geqq c$. *Then there exist* $N_0 \in \mathbf{N}$ *and* $b, \delta > 0$ *depending only on* Γ, Γ_0 *and* c, *such that if* $x0 \in \sigma$ *and* $y0 \in \tau$ *then*

$$d(x,y) \geqq \delta \max(|x|,|y|) - b \qquad \text{if } \max(|x|,|y|) \geqq N_0,$$

$$d(x,y) \geqq \delta |x|\,|y| - b \qquad \text{if } \min(|x|,|y|) \geqq N_0.$$

PROOF. Let γ be the geodesic arc joining x and y. Using the estimates of 2.3, the Euclidean size of γ is bounded below independently of σ, τ; for either one of $h(x)$, $h(y)$ is small, in which case γ is large; or x lies close to ξ and y to η so that the endpoints of γ are also close to ξ, η. Choose $\varepsilon > 0$ so that the Euclidean ε-ball centers ξ, η are disjoint and choose R so that $h(y) > R$ implies $y \in B_\varepsilon(\eta)$. Also choose R large enough that the circle C center 0 with hyperbolic radius R cuts all the above geodesics γ independently of σ, τ.

If $h(y) \leqq R$ then $y0$ lies inside C and $x0$ outside. Since $h(x,y)$ is measured along γ, $h(x,y) \geqq h(x,C) \geqq h(x) - R$.

If $h(y) > R$ then $y \in B_\varepsilon(\eta)$. Choosing $h(x)$ large enough ensures $x \in B_\varepsilon(\xi)$ also. Then $x0$, $y0$ both lie outside C on opposte sides of the cuts with γ, so $h(x,y) \geqq h(x) + h(y) - 2R$.

The result follows from 2.3.

LEMMA 3.5. *There exists* $q > 0$ *so that for* $p \geqq 1$, *if* $v \in V(e_1 \cdots e_p)$ *and* $w \in V(e_1 \cdots e_{p+q})$ *then* $d(v,w) \geqq N$.

PROOF. First consider the case where $\partial R(e_1 \cdots e_p) \cap D$ consists of the two extreme paths σ_1 and σ_2 from $e_1 \cdots e_{p-1}$ (cases 1 and 2a). Then $w \in \partial R(e_1 \cdots e_{p+q})$ lies on one of two or four possible extreme paths $\tau_1, \cdots, \tau_4$ in $G(\Gamma)$ all of whose endpoints lie in $S^1 \cap R(e_1 \cdots e_{p+m})$ provided $q \geqq m$.

An endpoint of such a path τ_i can only coincide with an endpoint of σ_j if we are in case 2a. Arguing as in the proof of Proposition 3.3, one sees that in this case $v \in V(e_1 \cdots e_p) \cap \sigma_j$ and $w \in V(e_1 \cdots e_{p+q}) \cap \tau_i$ implies $d(v,w) \geqq N$, provided we choose $q > N + m_1$.

Hence we may assume that the endpoints of τ_i, σ_j are distinct for each i, j. Apply $g = (e_1 \cdots e_{p-1})^{-1}$ to the whole picture. By the finiteness property 2.2 there are only a finite number of possible endpoints ξ_i of $g\sigma_i$ and only a finite number of possibilities for $g(\partial(S^1 \cap R(e_1 \cdots e_{p+m})))$. Thus $|\xi_i - \eta_j| \geqq c$ independent of q, p, i, j, where η_j, $j = 1, \cdots, 4$ is the endpoint of $g\tau_j$. Since $gv \in g\sigma_i$ and $gw \in g\tau_j$, by Lemma 3.4

$$d(v,w) = d(gv, gw) \geqq \delta \max(|gv|, |gw|) - b \quad \text{whenever } \max(|gv|,|gw|) \geqq N_0.$$

Since $w \in R(e_1 \cdots e_p)$ and $|w| \geqq p+q-1$ we have $|gw| \geqq q-1$. Therefore choosing q large enough gives the result.

It remains to consider the case 2b where $\partial R(e_1 \cdots e_p) \cap D$ consists of, say, the extreme right path σ_1 from $e_1 \cdots e_{p-1}$ and the extreme left path σ_2 from $f_1 \cdots f_{p-1}$, and $I(e_1 \cdots e_p)$ and $I(f_1 \cdots f_p)$ are adjacent intervals of rank p. Notice that $d(e_1 \cdots e_{p-1}, f_1 \cdots f_{p-1}) < m_0$. If $v \in \partial R(e_1 \cdots e_p)$ and $v \in \sigma \cup \tau$ then $d(v, e_1 \cdots e_{p-1}) < m$ also. Thus choosing $q > m + N$, we may assume that $v \in \sigma_1 \cup \sigma_2$.

Now as before, $w \in \partial R(e_1 \cdots e_{p+q})$ lies on one of four possible extreme paths $\tau_1, \cdots, \tau_4$ whose endpoints lie in $S^1 \cap R(e_1 \cdots e_{p+m})$ provided $q \geqq m$. Also in this case

$$S^1 \cap R(e_1 \cdots e_{p+m}) \subset \operatorname{Int}(S^1 \cap R(e_1 \cdots e_p)).$$

Applying $g = (e_1 \cdots e_{p-1})^{-1}$ to the picture, the finiteness property implies there are only a finite number of possible endpoints for intervals $gI(f_1 \cdots f_{p+m})$ with $d(e, gf_1 \cdots f_{p-1}) < m$. Likewise there are only a finite number of endpoints of intervals $gI(e_1 \cdots e_{p+m})$. The proof now works as before provided we note that if $w \in Z(f_1 \cdots f_{p-1})$,

$$\begin{aligned} |gw| = d(g^{-1}, w) &\geqq d(f_1 \cdots f_{p-1}, w) - d(g^{-1}, f_1 \cdots f_{p-1}) \\ &\geqq |w| - p + 1 - m, \end{aligned}$$

and that we always have $w \in Z(e_1 \cdots e_{p-1}) \cup Z(f_1 \cdots f_{p-1})$.

Generalised Proposition 3.3. *It is clear from the above that for general μ, Proposition 3.3 holds with m replaced by M.*

For general μ we may not have $\mu(g) > 0$ for all $g \in \Gamma_0$. However, since supp μ generates Γ, for each $g \in \Gamma_0$ there exists $k = k(g) = \min\{k : \mu^{(k)}(g) > 0\}$ where $\mu^{(k)}$ is the k-fold convolution of μ with itself. Let $K = \max\{k(g) : g \in \Gamma_0\}$. We note for future reference that in the above proof we may as well choose M so that $d(x, y) > K$ whenever $x \in \partial R(e_1 \cdots e_p)$ and $y \in \partial R(e_1 \cdots e_{p+M})$.

§4. The probabilities $u(x, y)$

Suppose $\{y_n\}_{n=1}^{\infty} \subset \Gamma$ and $\lim_{n \to \infty} y_n 0 = \xi \in \Lambda$. We shall obtain a formula

$$\frac{u(x, y_n)}{u(e, y_n)} = \frac{\langle v(x), T_q w_q \rangle}{\langle v(e), T_q w_q \rangle},$$

where $T_q = A_{i_1} \cdots A_{i_q}$ is a product of a finite number of certain possibly infinite strictly positive matrices $A_1, \cdots, A_k$; $v(x)$, $v(e)$ and w_q are strictly positive vectors, and $q \to \infty$ as $n \to \infty$.

Suppose $e_1 \cdots e_{(s+1)M} \in \Sigma_F$, where $s \geqq 1$ and M is as in Proposition 3.3. By the finiteness property 2.2, using the argument of Lemma 3.5, there are only a finite number of possible arrangements of pairs of sets

$$(e_1 \cdots e_{sM-1})^{-1} R(e_1 \cdots e_{sM}), (e_1 \cdots e_{sM-1})^{-1} R(e_1 \cdots e_{(s+1)M}).$$

One matrix $A = A_i$ is defined for each one of these possible pairs. The rows of A are indexed by $V_1 = (e_1 \cdots e_{sM-1})^{-1} V(e_1 \cdots e_{sM})$ and the columns by $V_2 = (e_1 \cdots e_{sM-1})^{-1} V(e_1 \cdots e_{(s+1)M})$. The (v, w) entry $a(v, w)$, $v \in V_1$, $w \in V_2$, is the probability that starting at v a random path first hits V_2 in w. We denote this matrix $A = A(e_1 \cdots e_{(s+1)M})$.

Lemma 4.1. *With the notation above, $a(v, w) > 0$ for all A, v, w.*

Proof. From the definition of $V(e_1 \cdots e_{sM})$ it is sufficient to prove that if $v, v' \in \partial R(e_1 \cdots e_{sM})$ then there is a path of positive probability from v to v' which does not meet $R(e_1 \cdots e_{(s+1)M})$.

If $\mu(g) > 0$ for each $g \in \Gamma_0$ we can clearly take the path joining v to v' in $\partial R(e_1 \cdots e_{sM})$.

More generally, one can join v to v' by a path of positive probability in which each vertex v'' is joined to a vertex in $\partial R(e_1 \cdots e_{sM})$ by a path of length at most K. By the remarks at the end of the previous section, such a path cannot meet $\partial R(e_1 \cdots e_{(s+1)M})$.

Suppose $A = A(e_1 \cdots e_{(s+1)M})$. Associated to A is a Hilbert space $H = H(e_1 \cdots e_{(s+1)M})$ with a complete orthonormal basis indexed by $(e_1 \cdots e_{sM-1})^{-1} V(e_1 \cdots e_{(s+1)M})$. H may be finite or infinite dimensional. Suppose $e_1 e_2 \cdots \in \Sigma$. We may associate to $e_1 e_2 \cdots$ the sequence of Hilbert spaces $H_{i_r} = H(e_1 \cdots e_{rM})$, $r = 1, 2, \cdots$ and the sequence of maps $A_{i_r} : H_{i_r} \to H_{i_{r-1}}$, $r \geqq 2$. (We shall show later that A_{i_r} is a bounded operator on H_{i_r}.)

Keeping the same fixed sequence $e_1 e_2 \cdots$, suppose $x, y \in \Gamma$. Let $V_t(x)$ be the row vector indexed by $V(e_1 \cdots e_{tM})$ whose vth entry is the probability that a path starting from x first hits $V(e_1 \cdots e_{tM})$ (if at all) in V. Let $w_r(y)$ be the column vector indexed by $V(e_1 \cdots e_{rM})$ whose wth entry is the probability that a path starting at $w \in V(e_1 \cdots e_{rM})$ ever reaches y.

Now suppose that $r > t > 0$ are such that $x \notin R(e_1 \cdots e_{tM})$ and $y \in R(e_1 \cdots e_{rM})$. Then any path from x to y must first hit $V(e_1 \cdots e_{tM})$ and then pass successively through $V(e_1 \cdots e_{(t+1)M}), \cdots, V(e_1 \cdots e_{rM})$ and finally jump to y.

By stationarity of the random walk, for any $g \in \Gamma$ the probability that a path starting at $v \in V(e_1 \cdots e_{sM})$ first reaches $V(e_1 \cdots e_{(s+1)M})$ in w is the same as the probability that a path starting at gv first hits $gV(e_1 \cdots e_{(s+1)M})$ in gw. Taking $g = (e_1 \cdots e_{sM-1})^{-1}$, we get that this probability is exactly the (v, w) entry in $A(e_1 \cdots e_{(s+1)M})$. Thus one obtains the formula

$$u(x, y) = \langle v_t(x), A_{i_{t+1}} A_{i_{t+2}} \cdots A_{i_r} w_r(y) \rangle.$$

Hence we have

Proposition 4.2. *Suppose $\{y_n\}_{n=1}^{\infty} \subset \Gamma$ and $\lim_{n\to\infty} y_n = \xi \in \Lambda$. Suppose also that $e_1 e_2 \cdots \in \pi^{-1}(\xi)$ and $x \in \Gamma$. Choose $t > 0$ such that $x \notin R(e_1 \cdots e_{tM})$. Then with the notation above, for any $r > t$,*

$$u(x, y_n) = \langle v_t(x), A_{i_{t+1}} \cdots A_{i_r} w_r(y_n) \rangle$$

for all sufficiently large n. (Here v_t, A_i, w_r all depend on the sequence $e_1 e_2 \cdots$.)

Proof. We only need note that since $y_n \to \xi = \pi(e_1 e_2 \cdots)$, for any $r > 0$ we have $y_n \in R(e_1 \cdots e_{rM})$ for all sufficiently large n.

Corollary 4.3. *In the situation of the Proposition, there exists a probability vector $v(e)$ so that*

$$\frac{u(x, y_n)}{u(e, y_n)} = \frac{\langle v(x), A_{i_t} \cdots A_{i_r} w_r(y_n) \rangle}{\langle v(e), A_{i_t} \cdots A_{i_r} w_r(y_n) \rangle} \qquad \textit{for large } n.$$

The proof is clear.

Remark 4.4. By replacing $v(x)$ by $v(x) A_{i_t}$ and w_r by $A_{i_r} w_r$ if necessary we may assume that all the coefficients of $v(x)$, $v(e)$ and $w_r(y_n)$ are strictly positive.

§5. Convergence of the Martin kernels

Our object in this section is to prove Theorems 1.1 and 1.2. The main idea is a version of the Perron–Frobenius theorem for the matrix products $A_{i_1} A_{i_2} \cdots$ of the last section. The proof of this theorem is inspired by [1]. We begin by showing that all of the matrices A_i are compact operators.

Lemma 5.1. *Let $x \in \Gamma$. There are constants $c > 0$ and $\lambda < 1$ so that*

$$u(e, x) \leqq c \lambda^{|x|}.$$

Proof. Let P be the convolution operator

$$Pf(x) = \sum_{g \in \Gamma} \mu(g) f(xg), \qquad x \in \Gamma, \quad f \in L^2(\Gamma).$$

By [4], P has spectral radius $\lambda < 1$; in other words $\overline{\lim}_{n\to\infty} \| P^n \|_2^{1/n} = \lambda < 1$. Thus $\| P^n \|_2 < \lambda^n$ for $n \geqq n_0$, say.

Let δ be the delta function at $e \in \Gamma : \delta(y) = 0$ if $y \neq e$ and $\delta(e) = 1$. Then

$$P^n \delta(y) = \sum_{z \in \Gamma} \mu^{(n)}(y^{-1}z)\delta(z) = \mu^{(n)}(y^{-1}),$$

$$\| P^n \delta \|_2^2 = \sum_{y \in \Gamma} (\mu^{(n)}(y^{-1}))^2 < \| P^n \|_2^2 \| \delta \|_2^2 < \lambda^{2n} \qquad \text{for } n \geqq n_0.$$

Hence $\mu^{(n)}(y^{-1}) < \lambda^n$ for $n \geqq n_0$.

Now $\mu^{(r)}(y) = 0$ whenever $r < |y|/N$ (recall $N = \max\{|g| : \mu(g) > 0\}$) and so

$$u(e, y) = \sum_{Nr \geqq |y|} \mu^{(r)}(y) < c\lambda^{[|y|N^{-1}]}.$$

Adjusting the constants gives the result.

LEMMA 5.2. *A_i is a compact operator for each i.*

PROOF. Obviously we need only consider those $A : H \to H'$ for which H' is infinite dimensional. It is enough to see that the image of the unit ball in H is contained in a Hilbert cube of the form

$$\left\{ h = \sum_{i,t} \lambda_{i_t} h_{i_t} : |\lambda_{i_t}| \leqq i^{-1} \right\} \quad \text{in } H',$$

where $\{h_{i_t} : i \in \mathbf{N}, 1 \leqq t \leqq q\}$ is an orthonormal basis of H'.

In the case $\mu(g) > 0$ if and only if $g \in \Gamma_0$, it is clear that for each $r \geqq 1$ there are at most four vertices v in the index set of the rows of A with $|v| = r$. Thus the rows may be indexed by $\mathbf{N} \times \{1, \cdots, 4\}$ in such a way that the (n, j)-component corresponds to a vertex v with $|v| \geqq n$; and similarly we can arrange to index columns so that the (n, j)-column corresponds to w with $|w| \geqq n$.

More generally there is an upper bound q, independent of r, to the number of vertices $|r|$ in the index sets of rows and columns with $|v| = r$. So indexing by $\mathbf{N} \times \{1, \cdots, q\}$ we may arrange that the (n, j)-vertex v always satisfies $|v| \geqq n$.

Suppose $A = A(e_1 \cdots e_{(s+1)M})$. Write

$$R = (e_1 \cdots e_{sM-1})^{-1} R(e_1 \cdots e_{sM}), \qquad R' = (e_1 \cdots e_{sM-1})^{-1} R(e_1 \cdots e_{(s+1)M}).$$

We have $R' \cap S^* \subset \mathrm{Int}(R \cap S^*)$. The vertices of the index set of the rows of A lie within a distance N of ∂R; likewise the vertices indexing the columns are within distance N of $\partial R'$. Therefore by Lemma 3.4 and the proof of 3.5, there are constants δ, b, N_0 so that a pair (v, w) indexing a coefficient of A satisfies

$$d(v,w) \geqq \delta \max(|v|,|w|) - b \qquad \text{if } \max(|v|,|w|) \geqq N_0,$$

$$d(v,w) \geqq \delta |v| \, |w| - b \qquad \text{if } \min(|v|,|w|) \geqq N_0.$$

Taking into account the method of indexing, this gives

$$\sum_{m,j} a^2_{(n,i):(m,j)} < \infty \qquad \text{for each } (n,i) \in \mathbf{N} \times \{1,\cdots,q\}$$

and

$$\sum_{m,j} a^2_{(n,i):(m,j)} < c'' e^{-\alpha n} \qquad \text{if } n \geqq N_0,$$

for some constant c'' independent of n.

These two conditions give the result.

We now copy the ideas found in [1].

Let H be one of the Hilbert spaces described above and let C be the positive cone $\{\Sigma_{i,t} \lambda_{i_t} h_{i_t} \in H : \lambda_{i_t} > 0 \text{ for all } i, t\}$. We can define a metric θ on the projective space $P(C)$ of lines in C by

$$\theta(f,g) = |\log[f, g, D, D']|,$$

where D, D' are the lines in which the plane spanned by f, g cuts ∂C and $[\cdot,\cdot,\cdot,\cdot]$ denotes cross-ratio (cf. the definition of hyperbolic distance in section 2). Suppose $A : H \to H'$ as above. Since all entries of A are strictly positive, $A(C) \subset C'$ where C' is the positive cone in H'.

Now θ is obviously continuous with respect to the topology induced on $P(C)$ from H (to see this rotate axes so that f, g lie in the plane spanned by the first two basis vectors of H), and similarly for θ'. Since by Lemma 5.2, A is a compact operator on H, $A(P(C))$ is compact in C' and so has finite θ'-diameter Δ say. By lemma 1 of [1],

$$\theta'(Af, Ag) \leqq \tanh\left(\frac{\Delta}{4}\right) \theta(f,g) \qquad \text{for } f, g \in P(C).$$

LEMMA 5.3. *Let $A_{i_1} A_{i_2} \cdots$ be the sequence of operators corresponding to some $e_1 e_2 \cdots \in \Sigma$, as explained in section 4. Let $T_n = A_{i_1} \cdots A_{i_n}$. Then*

$$T_n(C_{i_n}) \subset T_{n-1}(C_{i_{n-1}}) \qquad \textit{for each } n,$$

and there exist $c > 0$, $\beta < 1$ so that

$$\theta_{i_1}(T_n f, T_n g) \leqq c\beta^n \qquad \textit{whenever } f, g \in P(C_{i_n}).$$

PROOF. The first statement is immediate from the definitions and the second follows taking

$$\beta = \max\{\tanh \tfrac{1}{4}\Delta : \Delta \text{ is the } \theta' \text{ diameter of } A(C) \text{ for some } A\}.$$

PROOF OF THEOREM 1.1. Using the notation and result of Corollary 4.3, we must show that

$$\lim_{n\to\infty} \frac{\langle v(x), A_{i_1} \cdots A_{i_r} w_r(y_n)\rangle}{\langle v(e), A_{i_1} \cdots A_{i_r} w_r(y_n)\rangle} \quad \text{exists independently of } y_n.$$

Since $A_{i_r} w_r(y_n) \in C_{i_{r+1}}$, the result follows from Lemma 5.3.

PROOF OF THEOREM 1.2. (i) Choose a neighbourhood U of η with $\xi \notin U$. Suppose $e_1 e_2 \cdots \in \pi^{-1}(\xi)$, and suppose $z_n \in \Gamma$, $n = 1,2,\cdots$, and $\lim_{n\to\infty} z_n = \eta' \in U$. In the proof of Proposition 4.2 we can arrange that $z_n \notin R(e_1 \cdots e_{tM})$ whenever $z_n \in U$. By the Corollary to that Proposition and Theorem 1.1,

$$K(z_n, \xi) = \lim_{r\to\infty} \frac{\langle v(z_n), A_{i_1} \cdots A_{i_r} w_r\rangle}{\langle v(e), A_{i_1} \cdots A_{i_r} w_r\rangle} \quad \text{for any sequence } w_r \in C_{i_r}.$$

The components of $v(z_n)$ are probabilities that a path starting at z_n first enters $R(e_1 \cdots e_{tM})$ at some vertex w. By Lemma 3.4 it is clear that the length of any path from z_n to w is at least $\beta \mid z_n \mid - d$ for large n. By Lemma 5.1 therefore each component of $v(z_n)$ is less than or equal to $ce^{-\alpha|z_n|}$ for some constant c and $\alpha > 0$. Using Lemma 5.3 gives the result.

(ii) Consider $g_t = e_1 \cdots e_{tM+1}$ for $t \geqq 1$. Then g_t lies outside $R(e_1 \cdots e_{tM})$. Any random path from g_t to a point in $R(e_1 \cdots e_{tM})$ must hit a vertex in $V(e_1 \cdots e_{tM})$. Let $v(g_t)$ be the row vector indexed by $V(e_1 \cdots e_{tM})$ whose vth entry is the probability that a path starting at g_n first hits $V(e_1 \cdots e_{tM})$ in v. Arguing as in Proposition 4.2,

$$K(g_t, \xi) = \lim_{r\to\infty} \frac{\langle v(g_t), A_{i_1} \cdots A_{i_r} w_r\rangle}{\langle v(e), A_{i_1} \cdots A_{i_r} w_r\rangle} \quad \text{for some sequence } w_r \in C_{i_r}.$$

By the finiteness property 2.2 there are only a finite number of distinct vectors $v(g_t)$ for $t = 1,2,\cdots$. Let

$$\lim_{r\to\infty} \frac{A_{i_1} \cdots A_{i_r} w_r}{\|A_{i_1} \cdots A_{i_r} w_r\|} = w_0.$$

w_0 is independent of $\{w_r\}_{r=1}^{\infty}$. Therefore $\mathrm{Inf}_t \langle v(g_t), w_0\rangle > 0$.

On the other hand, the distance from 0 to $V(e_1 \cdots e_{tM})$ increases as $t \to \infty$ and so by Lemma 5.1 the components of $v(e)$ become exponentially small. Thus

$$K(g_t, \xi) = \frac{\langle v(g_t), w_0 \rangle}{\langle v(e), w_0 \rangle} \to +\infty.$$

Suppose now n is such that $tM \leqq n < (t+1)M$. Then for $z \in \Gamma$

$$u(e_1 \cdots e_n, z) \geqq u(e_1 \cdots e_n, g_t) u(g_t, z)$$

$$\geqq cu(g_t, z) \qquad \text{for some } c > 0, \quad \text{independent of } n.$$

Hence $K(e_1 \cdots e_n, z) \geqq cK(g_t, z)$ and so $\lim_{n\to\infty} K(e_1 \cdots e_n, \xi) = +\infty$ also.

§6. The harmonic measure ν

As described in the Introduction, the harmonic measure ν on Λ is defined by

$$\nu(E) = \Pr\left[\lim_{n\to\infty} x_1 \cdots x_n 0 \in E\right], \qquad E \subset \Lambda.$$

THEOREM 6.1. *ν is a Gibbs measure on Σ with respect to the function*

$$\phi(e_1 e_2 \cdots) = -\log K(e_1, e_1 e_2 \cdots).$$

PROOF. Let us first remark that Σ is not itself a subshift of finite type. However the finiteness property 2.2 allows one to code Σ almost bijectively to space Σ' which is of finite type. The details are carried out in [15]. Using the results of [15] §5 it is clear that it is enough to show that

(i) there are constants $c > 0$, $\eta < 1$ so that $|\phi(w) - \phi(w')| \leqq c\eta^n$ whenever $w, w' \in \Sigma$ and $w_i = w'_i$ for $i \leqq n$;

(ii) there are constants $T > 0$, P so that

$$T^{-1}\exp(S_n\phi(w) - Pn) < \nu(Z(w_1 \cdots w_n)) < T\exp(S_n\phi(w) - Pn), \quad n = 1, 2, \cdots,$$

whenever $w \in Z(w_1 \cdots w_n)$, where $S_n\phi(w) = \sum_{r=0}^{n-1} \phi(\sigma^r w)$ and σ is the left shift on Σ.

Suppose $w, w' \in \Sigma$ and $w_i = w'_i$ for $i \leqq n$. Then

$$|\phi(w) - \phi(w')| = \left|\log \frac{K(w_1, w)}{K(w_1, w')}\right|.$$

Suppose $rM \leqq n < (r+1)M$. By Corollary 4.3, there exists t independent of n so that

$$K(w_1, \pi(w)) = \lim_{m\to\infty} \frac{\langle v(w_1), A_{i_1} \cdots A_{i_t} A_{i_{t+1}} \cdots A_{i_m} w_m \rangle}{\langle v(e), A_{i_1} \cdots A_{i_t} A_{i_{t+1}} \cdots A_{i_m} w_m \rangle}$$

and

$$K(w_1, \pi(w')) = \lim_{m\to\infty} \frac{\langle v(w_1), A_{i_1} \cdots A_{i_r} A'_{i_{r+1}} A'_{i_m} w'_m \rangle}{\langle v(e), A_{i_1} \cdots A_{i_r} A'_{i_{r+1}} \cdots A'_{i_m} w'_m \rangle}$$

where $w_m, w'_m \in C_{i_m}$.

Let $u = A_{i_{r+1}} \cdots A_{i_m} w_m$, $u' = A'_{i_{r+1}} \cdots A'_{i_m} w'_m$, and $T = A_{i_1} \cdots A_{i_r}$. By Lemma 5.3, $\theta_{i_r}(Tu, Tu') \leqq c\beta^{r-t}$.

We wish to estimate the expression

$$\log\left(\frac{\langle v(w_1), Tu\rangle}{\langle v(e), Tu\rangle} \frac{\langle v(e), Tu'\rangle}{\langle v(w_1), Tu'\rangle}\right).$$

By Remark 4.4, we may assume $v(w_1)$, $v(e)$ and u, u' are strictly positive.

Rotate axes so that the first two basis vectors lie in the plane spanned by Tu, Tu' and let $Tu = (f_1, f_2, 0, \cdots)$, $Tu' = (g_1, g_2, 0, \cdots)$, $v(w_1) = (a, b, \cdots)$, $v(e) = (c, d, \cdots)$ in these co-ordinates. Let $\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We have to evaluate

$$\frac{af_1 + bf_2}{cf_1 + df_2} \cdot \frac{cg_1 + dg_2}{ag_1 + bg_2}.$$

According to [1],

$$\theta(Tu, Tu') = \left|\log \frac{f_2 g_1}{f_1 g_2}\right|$$

and

$$\theta(BTu, BTu') = \left|\log \frac{af_1 + bf_2}{cf_1 + df_2} \cdot \frac{cg_1 + dg_2}{ag_1 + bg_2}\right|$$

$$\leqq \tanh(\tfrac{1}{4}\lambda)\theta(Tu, Tu')$$

where $\lambda = |\log(ad/bc)|$.

Since B is independent of m and r we obtain

$$|\phi(w) - \phi(w')| \leqq c'\eta^n \qquad \text{for some } c' > 0, \quad \eta < 1.$$

(ii) Observe that $K(x, w): \Gamma \times \Lambda \to \mathbf{R}$ satisfies the cocycle identity

$$K(x, \xi)K(y, x^{-1}\xi) = K(xy, \xi), \qquad x, y \in \Gamma, \quad \xi \in \Sigma.$$

(Actually $\sigma(x, w) = K(x^{-1}, \xi)$ is a cocycle in the usual sense.)

This follows by taking limits in the relation

$$\frac{u(x, z)}{u(e, z)} \cdot \frac{u(y, x^{-1}z)}{u(e, x^{-1}z)} = \frac{u(xy, z)}{u(e, z)}, \qquad x, y, z \in \Gamma.$$

It follows that $S_n\phi(\xi) = -\log K(e_1 \cdots e_{n-1}, \sigma^{n-1}\xi)$, and

$$\exp(S_n\phi(\xi)) = \lim_{r\to\infty} \frac{\langle v_n(e), A_{i_1} \cdots A_{i_r} w_r\rangle}{\langle v(e_1 \cdots e_{n-1}), A_{i_1} \cdots A_{i_r} w_r\rangle}$$

where t is chosen so that $M(t-1) \leqq n < Mt$, $e_1 e_2 \cdots \in \pi^{-1}(\xi)$, and $w_r \in C_{i_r}$.

On the other hand, $\nu(Z(e_1 \cdots e_n))$ is the probability that a path starting at 0 converges to a point $\xi \in Z(e_1 \cdots e_n)$. By reasoning as in Proposition 3.3 one sees that any such path eventually hits $V(e_1 \cdots e_{M(t-1)})$.

Let u_t be the column vector with entries indexed by $V(e_1 \cdots e_{M(t-1)})$, whose vth entry is the probability that starting at v a path converges to $\eta \in Z(e_1 \cdots e_n)$. Then

$$\nu(Z(e_1 \cdots e_n)) = \langle w_n(e), A_{i_{t-1}} u_t \rangle,$$

where $w_n(e)$ is the row vector indexed by $V(e_1 \cdots e_{M(t-1)})$ whose wth entry is the probability that, starting at e, a path first hits $V(e_1 \cdots e_{(t-1)M})$ in w.

Notice that $v_n(e) = w_n(e) A_{i_{t-1}}$.

Writing $q_r = A_{i_t} A_{i_{t+1}} \cdots A_{i_r} w_r$ we obtain

$$\frac{\exp S_n \phi(\xi)}{\nu(Z(e_1 \cdots e_n))} = \lim_{r \to \infty} \frac{\langle w_n(e), A_{i_{t-1}} q_r \rangle}{\langle w_n(e), A_{i_{t-1}} u_t \rangle \langle v((e_1 \cdots e_{n-1})^{-1}), q_r \rangle}.$$

The vectors $p_r = q_r / \| q_r \|$ lie in a compact set in the unit ball of $H_{i_{t-1}}$ since they are in the image of A_{i_t}. Since $e_1 \cdots e_n$ is at a bounded distance from $R(e_1 \cdots e_{Mt})$, by the finiteness property there are only a finite number of different vectors $v(e_1 \cdots e_{n-1})$, independent of n. Therefore

$$0 < \inf_{r,n} \langle v(e_1 \cdots e_{n-1}) p_r \rangle \leqq \sup_{r,n} \langle v(e_1 \cdots e_{n-1}), p_r \rangle < \infty.$$

Again by the finiteness property, there are only a finite number of possible vectors u_t. The vectors $A_{i_{t-1}} p_r$ and $A_{i_{t-2}} u_t$ therefore satisfy

$$0 < M_2 \leqq \| A_{i_{t-1}} p_r \|, \| A_{i_{t-1}} u_t \| \leqq M_1 < \infty$$

for constants M_1, M_2 independent of t and r.

Take an orthonormal basis in the plane spanned by $A_{i_{t-1}} p_r$, $A_{i_{t-1}} u_t$, so that with respect to the co-ordinates in this plane, $A_{i_{t-1}} p_r = (a, b)$ and $A_{i_{t-1}} u_t = (c, d)$ are strictly positive.

Let

$$K = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \alpha, \beta, \gamma, \delta \geqq 0,\ M_1^2 \leqq \alpha^2 + \beta^2,\ \gamma^2 + \delta^2 \leqq M_1^2 \right\}.$$

Then K is compact and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K$. Moreover the image under K of any interval $[T_1, T_2]$, $T_1, T_2 > 0$, is again an interval bounded away from zero and infinity.

Let the components of $w_n(e)$ with respect to the two basis vectors above be (λ_1, λ_2). We may arrange that $\lambda_1, \lambda_2 > 0$. Then

$$\frac{\langle w_n(e), A_{i_{t-1}} p_t \rangle}{\langle w_n(e), A_{i_{t-1}} u_t \rangle} = \frac{a\lambda_1 + b\lambda_2}{c\lambda_1 + d\lambda_2}.$$

Now $w_n(e) = z_n(e)A_{i_{t-2}}$, where $z_n(e)$ is the row vector indexed by $V(e_1 \cdots e_{M(t-2)})$ whose wth entry is the probability that starting at e, a random path first hits $V(e_1 \cdots e_{M(t-2)})$ in w. The proof of Lemma 5.2 shows equally that A_i^T is a compact operator for each i, hence the image of $z_n(e)A_{i_{t-2}}$ lies in a compact set in the interior of the positive cone of the corresponding projective space independently of n and $e_1e_2\cdots$.

Therefore λ_1/λ_2 is bounded uniformly away from zero and infinity. Combining this with the remarks about K above, one has that $(a\lambda_1 + b\lambda_2)/(c\lambda_1 + d\lambda_2)$ is uniformly bounded away from zero and infinity, which is the result we require.

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Mathematics Institute
University of Warwick
Coventry CV4 7AL, England